
Chaotic dynamics

an introduction

SECOND EDITION

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Introduction

The irregular and unpredictable time evolution of many nonlinear systems has been dubbed ‘chaos.’ It occurs in mechanical oscillators such as pendula or vibrating objects, in rotating or heated fluids, in laser cavities, and in some chemical reactions. Its central characteristic is that the system does not repeat its past behavior (even approximately). Periodic and chaotic behavior are contrasted in Figure 1.1. Yet, despite their lack of regularity, chaotic dynamical systems follow deterministic equations such as those derived from Newton’s second law.

The unique character of chaotic dynamics may be seen most clearly by imagining the system to be started twice, but from slightly different initial conditions. We can think of this small initial difference as resulting from measurement error, for example. For nonchaotic systems this uncertainty leads only to an error in prediction that grows *linearly* with time. For chaotic systems, on the other hand, the error grows *exponentially* in time, so that the state of the system is essentially unknown after a very short time. This phenomenon, which occurs only when the governing equations are nonlinear, is known as *sensitivity to initial conditions*. Henri Poincaré (1854–1912), a prominent mathematician and theoretical astronomer who studied dynamical systems, was the first to recognize this phenomenon. He described it as follows: ‘... it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon’ (Poincaré, 1913).

If prediction becomes impossible, it is evident that a chaotic system can resemble a stochastic system (a system subject to random external forces). However, the source of the irregularity is quite different. For

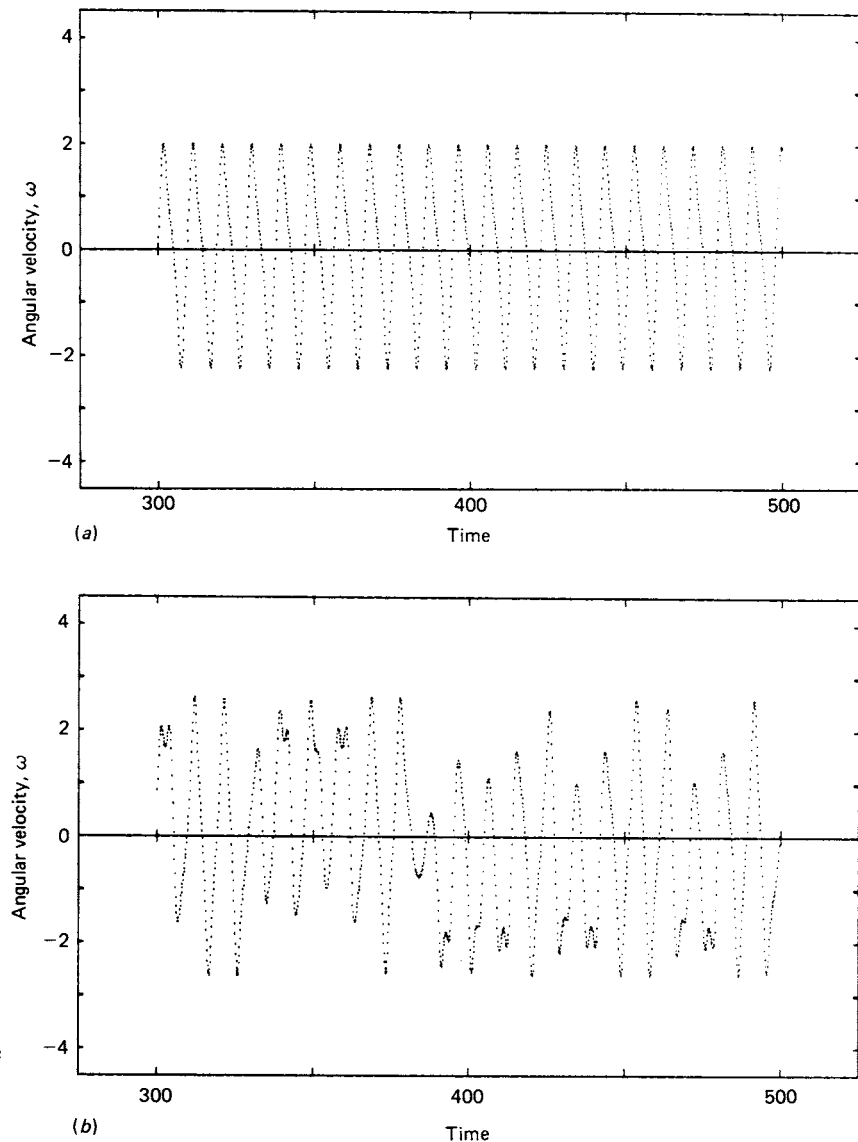


Fig. 1.1 The damped, driven pendulum can exhibit both periodic and chaotic motions. Here, the angular velocity is shown as a function of time for the two cases.

chaos, the irregularity is part of the intrinsic dynamics of the system, not unpredictable outside influences.

Chaotic motion is not a rare phenomenon. Consider a dynamical system described by a set of first order differential equations. Necessary conditions for chaotic motion are that (a) the system has at least three independent dynamical variables, and (b) the equations of motion contain a nonlinear term that couples several of the variables. The equations can often be expressed in the form:

$$\left. \begin{aligned} dx_1/dt &= F_1(x_1, x_2, \dots, x_n), \\ dx_2/dt &= F_2(x_1, x_2, \dots, x_n), \\ &\vdots \\ dx_n/dt &= F_n(x_1, x_2, \dots, x_n), \end{aligned} \right\} \quad (1.1)$$

where n must be at least 3. Two examples of appropriate nonlinear equations are:

$$\left. \begin{aligned} dx_1/dt &= \alpha x_1 + \beta x_2 + \gamma x_1 x_2 + \dots + \delta x_n, \\ dx_1/dt &= \alpha x_1 + \beta x_2 + \gamma \sin x_2 + \dots + \delta x_n \end{aligned} \right\} \quad (1.2)$$

where $\alpha, \beta, \gamma, \delta$ are constants. In each case the nonlinear term couples both x_1 and x_2 . Systems such as these are often chaotic for some choices of the constants.

The fact that only three variables are required for chaos was surprising when first discovered. We shall see that three-space is sufficient to allow for (a) divergence of trajectories, (b) confinement of the motion to a finite region of the phase space of the dynamical variables, and (c) uniqueness of the trajectory. The nonlinearity condition is perhaps less surprising. Solutions to linear differential equations can always be expressed as a linear superposition of periodic functions, once initial transients have decayed. The effect of a nonlinear term is often to render a periodic solution unstable for certain parameter choices. While these conditions do not guarantee chaos, they do make its existence possible.

The nonlinearity condition has probably been responsible for the late historical development of the study of chaotic systems. Despite the fact that chaotic systems are deterministic and are described by many of the long-known classical equations of physics, the development of the subject itself is more recent. This circumstance may arise from the fact that, with the exception of some first order equations, nonlinear differential equations are either difficult or impossible to solve analytically. Although it is sometimes possible to use linearized approximations, the solution of nonlinear differential equations generally requires numerical methods whose practical implementation demands the use of a digital computer. The first numerical study to detect chaos in a nonlinear dynamical system was that of Lorenz's model of convective fluid flow (Lorenz, 1963). Similarly, the majority of the diagrams in this book are based upon the use of numerical methods on a personal computer to solve nonlinear equations.

From these general comments on chaotic systems, we turn to the physical system that is the focus of this work – the damped, driven pendulum. The choice of the pendulum as a model system has strong

historical precedent in physics. Galileo postulated the constancy of period for small amplitude oscillations of the pendulum from observations of swaying lamps in the cathedral at Pisa in 1581 (Robinson, 1921). He took up the problem of the relationship between the period and pendulum length in his famous *Dialogue on the Two Principal World Systems* in 1632, and in 1637 he suggested that the square of the period was proportional to the length of the pendulum for small oscillation amplitudes (Dugas, 1958). The pendulum also served as a primary timing mechanism for clocks and as a method of measuring variations in the earth's gravitational field. As a pedagogical device the pendulum has long been a standard mechanical example in introductory physics and classical mechanics courses. Now, 400 years after Galileo's initial work, the pendulum has again become an object of research as a chaotic system. The references scattered throughout this work attest to its popularity.

The damped, sinusoidally driven pendulum of mass m (or weight W) and length l is described by the following equation of motion:

$$ml^2 \frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + Wl \sin \theta = A \cos(\omega_D t). \quad (1.3)$$

This equation expresses Newton's second law with the various terms on the left representing acceleration, damping, and gravitation. The angular velocity of the forcing, ω_D , may be different from the natural frequency of the pendulum. In order to minimize the number of adjustable parameters the equation may be rewritten in dimensionless form as:

$$d^2\theta/dt^2 + (1/q)d\theta/dt + \sin\theta = g \cos(\omega_D t) \quad (1.4)$$

where q is the damping or quality parameter, g is the forcing amplitude, *not to be confused with the gravitational acceleration*, and ω_D is the drive frequency. The low-amplitude natural angular frequency of the pendulum is unity, and time is regarded as dimensionless. (This particular notation follows that used by Gwinn and Westervelt. See, for example, Gwinn and Westervelt (1986).) This equation satisfies the necessary conditions for chaos when it is written as a set of first order equations:

$$\left. \begin{aligned} d\omega/dt &= -(1/q)\omega - \sin\theta + g \cos\phi, \\ d\theta/dt &= \omega, \\ d\phi/dt &= \omega_D. \end{aligned} \right\} \quad (1.5)$$

The variable ϕ is introduced as the phase of the drive term. The

necessary three variables (ω, θ, ϕ) are evident, and the $\sin\theta$ and $g\cos\phi$ terms are clearly nonlinear. Whether the motion is chaotic depends upon the values of the parameters g , ω_D , and q . For some values the pendulum locks onto the driving force, oscillating in a periodic motion whose frequency is the driving frequency, possibly with some harmonics or subharmonics. But for other choices of the parameters the pendulum motion is chaotic. One may view the chaos as resulting from a subtle interplay between the tendency of the pendulum to oscillate at its ‘natural’ frequency and the action of the forcing term. The transitions between nonchaotic and chaotic states, due to changes in the parameters, occur in several ways and depend delicately upon the values of the parameters.

A variety of analytic and computational tools may be used in the study of chaotic systems. In Chapter 2 several of these are discussed. The pendulum’s phase space and its properties are described, together with the conceptual device known as the Poincaré section. Then, since Fourier spectra are an indicator of chaotic motion, some elements of Fourier analysis are outlined. Chapter 3 is a description of the application of these and other techniques to the pendulum.

The driven pendulum would seem to be one of the simplest physical systems. Yet its behavior is rich and complex. The study of its motion can be facilitated by simple mathematical models formulated as difference equations, that provide a discrete *mapping* of the system from one state to another. Mappings have the advantage of being conceptually simple and numerically efficient, and they may be used as paradigms for various aspects of the pendulum motion. Chapter 4 contains discussions of three such maps, the logistic map, the circle map, and the horseshoe map. We use them to provide insight into the behavior of the pendulum.

Chapter 5 is concerned with the geometric structure of the *attractor* that describes the chaotic pendulum. The attractor, and its Poincaré section, are *fractal* structures with noninteger dimensionality. Various approaches to the calculation of fractal dimension are described. Another geometric feature is the exponential divergence of the chaotic trajectories on the attractor. The rate of this divergence is characterized by Lyapunov exponents. The calculation of these exponents and their relation to (a) the fractal dimension, (b) the dissipative nature of the pendulum, and (c) the duration of predictable behavior are also discussed.

Up to this point the presentation is focused on the fundamental ideas of chaotic dynamics. In Chapter 6 we discuss the relationship between these ideas and the analysis of experimental data. The developing methodology for characterization of nonlinear dynamical behavior in

experimental phenomena is complex. In this chapter we describe some of these methods and apply them to experimental data from a physical pendulum. The results of this study are then compared to those from the numerical simulations developed earlier in the book. The experimental data are also used to illustrate the possibility of prediction of chaotic data. Finally, we illustrate, through numerical simulation, the control of unstable dynamical states, in an otherwise chaotic pendulum.

Chapter 7 concludes the book with a brief survey of chaotic behavior in physical systems, including lasers, chemical reactions, fluids, crystal growth, and earthquakes. We emphasize the extension of chaotic dynamics to spatially extended systems having many degrees of freedom. Finally, the application of chaotic dynamics to quantum systems, and the connection between chaos and irreversibility are also discussed briefly.

Two appendices present numerical aspects of this book. Appendix A is a description of the Runge–Kutta algorithm used to solve the pendulum differential equation. Appendix B provides brief descriptions and listings of the computer programs used throughout the text, and in the computer exercises given at the end of several of the chapters. The listings utilize the language True BASIC™, but they are adaptable to any compiled BASIC or other high level language. (Interpreted BASIC, which is typically delivered with current microcomputers, is too slow for most of these simulations. The exceptions are the mappings in Chapter 4.) A third appendix, C, provides solutions to selected problems.